Price of Anarchy of Traffic Assignment with Exponential Cost Functions

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About Me

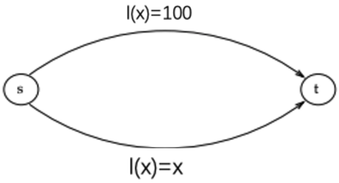
- Bachelor of Engineering (Electrical Engineering (Computer)) at the University of Sydney (2013-2017)
- Doctor of Philosophy at Western Sydney University & IIIA-CSIC & Universitat Autònoma de Barcelona (2018-2023)
  Thesis title: Smart Traffic Control for the Era of Autonomous Driving
- Visiting student at the University of Oxford supervised by Michael Wooldridge.
- Research Assistant at the University of Wollongong supervised by Bo Du.
- Research Fellow at the University of South Australia.

Motivation

- Individual decision-making by drivers usually shows the selfish behaviour of agents and the network performance drops due to selfish behaviour (called distributed control).
- The advent of V2X technology provides the possibility of centralized control.
- Centralized control or distributed control?
- The Price of Anarchy (PoA) is an index to qualify selfish decision-making in the worst-case scenario.
- We claim that POA can be used as a threshold to switch between centralized or distributed control.

Traffic Assignment Problem [Roughgarden,2003]

- A road network $G = (V, E)$ is a directed graph, where:
  - V is a set of positions;
  - E is a set of roads;
- An approach of the congestion game [ROBERT W. 1973];
- Origin-destination: a pair of positions;
- Demand: a number of vehicles;
- Routes: a sequence of roads linked between an origin-destination;
- Traffic flow: a number of vehicles on each route;
- Cost function: $l_e(x)$ is travel time of vehicles driving on road e, where x is the number of vehicles on the road;



Number of Vehicles: 100

Traffic flow:
Top route: 60 vehicles
Bottom route: 40 vehicles

Travel time:
Top route: 100s
Bottom route: 40s

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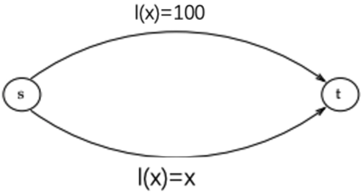


Number of Vehicles: 100

NE Traffic flow:
Top route: 0 vehicles
Bottom route: 100 vehicles

Travel time:
Top route: 100s
Bottom route: 100s

Curving Fitting Results with Real-world Data

- Most common cost function is BPR function (US Government):

$$l(x) = ax^b + c$$

- Akcelik function (Australian Government):

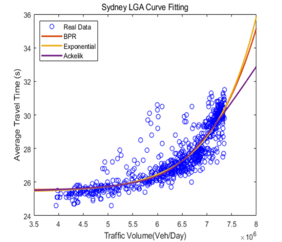
$$l(x) = c + \frac{3600}{4} a[(x-1) + \sqrt{(x-1)^2 + \frac{8bx}{da}}]$$

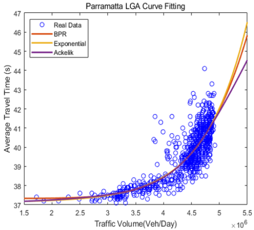
- We proposed a new cost function (variation from BPR):

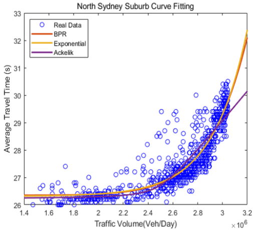
$$l(x) = ae^{bx} + c$$

Curving Fitting Results with Real-world Data

LGA Map







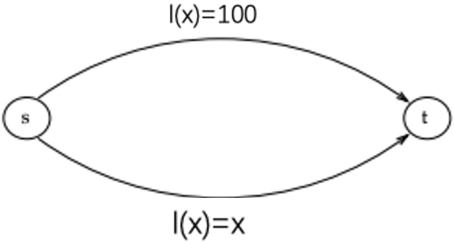
Curving Fitting Results with Real-world Data

Region	Model	a	b	c	R^2
Sydney	BPR	1.729	8.269	25.54	0.8005
	Akcelik	0.01667	216.7	25.37	0.7957
	Exponential	0.001047	7.481	25.41	0.8023
Parramatta	BPR	2.361	6.386	37.33	0.7055
	Akcelik	0.01586	304.1	37.06	0.7049
	Exponential	0.007255	5.863	37.13	0.7058
North Sydney	BPR	0.4403	10.41	26.35	0.8205
	Akcelik	0.00762	11.9	26.23	0.7859
	Exponential	7.069e-05	8.876	26.31	0.8219

Price of Anarchy [Elias, 1999; Roughgarden,2003]

- Road Network Instance: (G, r, L)
- Social cost is the total travel time of all vehicles
- $C(NE)$: Social cost of the Nash equilibrium flow
- $C(SO)$: Social cost of the Global optimization flow
- The price of anarchy of a road instance

$$PoA(G, r, L) = \frac{C(NE)}{C(SO)}$$



Number of Vehicles: 100

NE: (100,0)
C(NE)=100*100=10000s

SO: (50,50)
C(SO)=50*100+50*50=7500s

PoA=4/3

Price of Anarchy [Roughgarden,2003]

The price of anarchy of a class of traffic assignment problems is the supremum of the price of anarchy of all the problems in the class.

Description	Typical Representative	Price of Anarchy
Linear	$ax + b$	$\frac{4}{3} \approx 1.333$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.626$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[4]{4}}{4\sqrt[4]{4}-3} \approx 1.896$
Polynomials of degree $\leq p$	$\sum_{i=0}^{p} a_i x^i$	$[1 - p \cdot (p+1)^{\frac{-1-p}{p}}]^{-1} = \Theta(\frac{p}{lnp})$
M/M/1 delay functions	$(u - x)^{-1}$	$\frac{1}{2}(1 + \sqrt{\frac{u_{min}}{u_{min} - R_{max}}})$
Exponential	?	?

BPR: PoA=2.394 (p=5) PoA=3.512 (p=10)

Sydney CBD: PoA = 3.1 (p = 8.12)

PoA of Exponential Function

Theorem

For any instance (G, r, L_{exp}) with exponential cost functions, the price of anarchy satisfies the following.

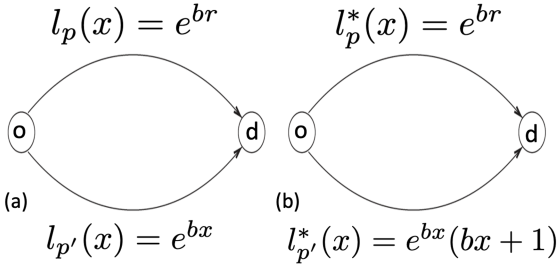
$$PoA(G, r, L_{exp}) \leq \frac{br}{br + 2 - W(e^{br+1}) - \frac{1}{W(e^{br+1})}} \tag{1}$$

where $W(\cdot)$ is the Lambert W function.

PoA of Exponential Function

Theorem

For any positive numbers b and r, there exists an instance (G, r, L_{exp}) with exponential cost functions, for which $PoA(G, r, L_{exp})$ is exactly equal to $\frac{br}{br+2-W(e^{br+1})-\frac{1}{W(e^{br+1})}}$.



PoA of Exponential Function

For any $x \geq e$ [Mehdi Hassani, 2007]:

$$log(x) - loglog(x) \quad \leq \quad W(x) \quad \leq \quad log(x) - \frac{1}{2} loglog(x) \qquad (2)$$

Lemma

For any non-negative x, we have

$$\frac{x}{log(x+1)} \leq \frac{x}{x+2-W(e^{x+1})-\frac{1}{W(e^{x+1})}} \leq \frac{2x}{log(x+1)}$$

Conjecture

Given a road network G and any traffic demand r, we have (G, r, L_{BPR}) and (G, r, L_{exp}). If $\Phi \cdot \hat{b} \leq \hat{n}$ and $f_p^* \leq \Phi$ for all $p \in P$ (where f^* is the equilibrium flow of the instance with exponential functions), then $PoA(G, r, L_{exp}) \leq \hat{PoA}_{BPR}$.

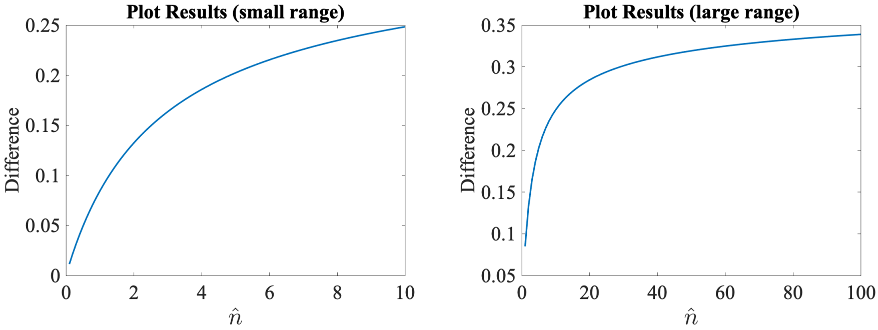


Figure: Plot Results for Conjecture

PoA of Exponential Function

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M/M/1 delay functions	$(u - x)^{-1}$	$\frac{1}{2}(1 + \sqrt{\frac{u_{min}}{u_{min} - R_{max}}})$
Exponential	$ae^{bx} + c$	$\frac{br}{br+2-W(e^{br+1})-\frac{1}{W(e^{br+1})}} \approx \frac{2br}{log(br+1)}$

Conclusion

- Proposed an exponential cost function with the support of real-world data.
- Proved tight upper bound of *PoA* when using the proposed exponential function.
- Compare the expression of *PoA* for the BPR function and for the exponential function.


THANK YOU!